Tutorial 11.

Preliminary:

① Term Structure. t-year

The yield to maturity on a zero coupon bond is called the spot rate of interest $S_0(t)$.

The set $\{S_0(t)\}_{t>0}$ is the term structure of interest rate.

② Relationship between spot rates and yield to maturity

If the face amount of the bond is F, coupon rate is r, yield to maturity y_r has

$$P = Fr[(1+y_r)^{-1} + \cdots + (1+y_r)^{-k}] + (F+Fr)(1+y_r)^{-(k+1)}$$

$$= Fr[(1+S_0(1))^{-1} + (1+S_0(2))^{-2} + \cdots + (1+S_0(k))^{-k}] + (F+Fr)(1+S_0(k+1))^{-(k+1)}.$$

③ Forward rate: Given the term structure at time, then $n-1$ year forward, one-year rate from $n-1$ to n,

$$i_0(n-1,n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}} - 1.$$

Exercise:

6.1.4.

(a) $$P = Fr \cdot \sum_{k=1}^{n} (1+S_0(k))^{-k} + F(1+S_0(n))^{-n}.$$

Bond 1: $r = \frac{4\%}{2} = 2\%$, $P = 85.12$, then

$$85.12 = 0.02 \times 100 \sum_{k=1}^{20} (1+S_0(k))^{-k} + 100(1+S_0(20))^{-20} \quad \cdots ①$$

Bond 2: $r = \frac{10\%}{2} = 5\%$, $P = 133.34$, then

$$133.34 = 0.05 \times 100 \sum_{k=1}^{20} (1+S_0(k))^{-k} + 100(1+S_0(20))^{-20} \quad \cdots ②$$

$\Rightarrow S_0(20) = 0.0323$

Yield rate 10-year zero coupon bond is $2 \times S_0(20) = 0.0646$.

(b) Assume $F=1$, then

$$P = r\sum_{k=1}^{n}(1+S_0(k))^{-k} + (1+S_0(n))^{-n}.$$

For yield j,

$$P = (1+j)^{-n} + r\sum_{k=1}^{n}(1+j)^{-k}, \text{ then}$$

Set a function $f(x) = (1+x)^{-n} + r\sum_{k=1}^{n}(1+x)^{-k}$ which is monotonically decreasing with x.

since $S_0(1) = S_0(2) = \cdots = S_0(n-1) < S_0(n)$, then $P = r\sum_{k=1}^{n}(1+S_0(1))^{-k} + (1+S_0(n))^{-n} < r\sum_{k=1}^{n}(1+S_0(n-1))^{-k} + (1+S_0(1))^{-n}$

$> r\sum_{k=1}^{n}(1+S_0(n))^{-k} + (1+S_0(n-1))^{-n}$

$f(S_0(n-1)) > P = f(j) > f(S_0(n))$. $\Rightarrow$ $S_0(n-1) < j < S_0(n)$.

6.1.5.

For ½ year : $S_0(1) = 0.05$ because $P = \dfrac{1+4\%/2}{1+\frac{0.05}{2}} = \dfrac{1+4\%/2}{1+\frac{S_0(1)}{2}}$.

For 1 year : $P = \dfrac{6\%/2}{1+\frac{0.10}{2}} + \dfrac{1+6\%/2}{(1+\frac{0.10}{2})^2} = \dfrac{6\%/2}{1+\frac{S_0(1)}{2}} + \dfrac{1+6\%/2}{(1+\frac{S_0(2)}{2})^2}$

$\Rightarrow S_0(2) = 0.1008$

For 1½ year : $P = \dfrac{4\%/2}{1+\frac{0.15}{2}} + \dfrac{4\%/2}{(1+\frac{0.15}{2})^2} + \dfrac{4\%/2+1}{(1+\frac{0.15}{2})^3} = \dfrac{4\%/2}{1+\frac{S_0(1)}{2}} + \dfrac{4\%/2}{(1+\frac{S_0(2)}{2})^2} + \dfrac{4\%/2+1}{(1+\frac{S_0(3)}{2})^3}$

$\Rightarrow S_0(3) = 0.15151$.

For 2 year : $P = \dfrac{8\%/2}{1+\frac{0.15}{2}} + \cdots + \dfrac{8\%/2+1}{(1+\frac{0.15}{2})^4} = \dfrac{8\%/2}{(1+\frac{S_0(1)}{2})} + \dfrac{8\%/2}{(1+\frac{S_0(2)}{2})^2} + \dfrac{8\%/2}{(1+\frac{S_0(3)}{2})^3} + \dfrac{1+8\%/2}{(1+\frac{S_0(4)}{2})^4}$

$\Rightarrow S_0(4) = 0.15234$.

Hence, the term structure set $\{S_0(1), S_0(2), S_0(3), S_0(4)\} = \{0.05, 0.1008, 0.15151, 0.15234\}$.

6.4.4.

(a) $S_0(1) = \frac{8\%}{2} = 0.04$, $S_0(2) = \frac{10\%}{2} = 0.05$, $S_0(3) = \frac{x\%}{2}$. $y_r = \frac{11\%}{2} = 0.055$, $r = \frac{10\%}{2} = 0.05$.

$P = 1.05\times(1+S_0(3))^{-3} + 0.05\times[(1+S_0(1))^{-1} + (1+S_0(2))^{-2}] = 1.05\times(1+y_r)^{-3} + 0.05\times[(1+y_r)^{-1} + (1+y_r)^{-2}]$

$\Rightarrow x\% = 11.09\% = S_0(3)$.

(b) $i_0(2,3) = \dfrac{(1+S_0(3))^3}{(1+S_0(2))^2} - 1 = \dfrac{(1+\frac{x\%}{2})^3}{(1+0.05)^2} - 1 = 0.11 \Rightarrow x\% = 10.33\%$.

(c)

0	$\frac{1}{2}$	1
get 0.9615	borrow 1	get 1.06 pay back 1.05 profit = 0.01

0 : sell a 6-month bond get $\frac{1}{1.04} = 0.9615$.
buy a 1-year bond

$\frac{1}{2}$: borrow 1 with interest rate 10%.

1 : get $0.9615\times(1+\frac{10\%}{2})^2 = 1.06$ from bond, pay back $1+\frac{0.1}{2} = 1.05$.